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ON THE INVERSION OF SINGULAR INTEGRAL EQUATIONS TO WHICH CERTAIN PARTICULAR PROBLEMS OF POTENTIAL THEORY REDUCE*

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This paper is devoted to the inversion and study of certain singular integral equations of a special kind (together with a moving singularity, a fixed singularity is also present). Equations of the structure under consideration occur in examining torsion problems of homogeneous and piecewise-homogeneous rods. Explicit representations are obtained for the solution of the equations, and constraints are set up on the right-hand side for which these solutions are valid.

1. We consider singular integral equations of the form

$$\frac{2}{\pi i}\int \delta(t) K^{\mp}(t, t_0, \rho)\, dt = g(t_0), \quad -\rho < t_0 < \rho \tag{1.1}$$
$$K^{\mp}(t, t_0, \rho) = \frac{1}{t-t_0} \mp \frac{\rho}{tt_0-\rho^2}$$

Here and henceforth, unless stipulated otherwise, the integration will be over a rectilinear segment $[-\rho, \rho]$; $\delta(t)$ is the desired density and $g(t_0)$ is a given function that is Hölder-continuous in this same segment. The inversion formulas corresponding to these equations are

$$\delta(t_0) = \frac{1}{2\pi i}\int g(t) K^{\pm}(t, t_0, \rho)\, dt \tag{1.2}$$

That these formulas yield an authentic inversion of the corresponding Eqs.(1.1) can be seen by direct substitution, by inserting the indicated values of $\delta(t_0)$ under the integral sign in (1.1) and then performing certain assumed operations. The basis of the proof is the process of evaluating the singular integrals (the actual construction of the Poincaré-Bertrand formulas for discontinuous contours).

2. We turn to an examination of a singular integral equation of the second kind

$$\mu(t_0) - \lambda\int \mu(t) K^{+}(t, t_0, \rho)\, dt = f(t_0) \tag{2.1}$$

where λ is a certain parameter, $f(t_0)$ is a given Hölder function, and $\mu(t_0)$ is the density to be determined. Setting therein

$$g(t_0) = \frac{1}{\lambda}[\mu(t_0) - f(t_0)]$$

and inverting the equation obtained according to (1.2), we obtain

$$\mu(t_0) + \frac{1}{\pi^2\lambda}\int \mu(t) K^{-}(t, t_0, \rho)\, dt = F(t_0) \tag{2.2}$$
$$F(t_0) = \frac{1}{\pi^2\lambda}\int f(t) K^{-}(t, t_0, \rho)\, dt$$

Multiplying (2.2) by $\pi^2\lambda^2$ and subtracting it term by term from the initial Eq.(2.1), we arrive at the following singular equation:

$$(\pi^2\lambda^2 - 1)\mu(t_0) + 2\lambda\int \mu(t)\frac{dt}{t-t_0} = R(t_0) \tag{2.3}$$
$$R(t) = \pi^2\lambda^2 F(t) - f(t)$$

The solution of (2.3) can be obtained by a well-known method /1, 2/.

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Taken in the more complicated Tricomi form, the singular Eq.(2.1) was studied earlier by Tricomi himself, and then by Mikhlin and Bitsadze /1, 3/.

We note that a singular equation of the form

$$\mu(t_0) - \lambda \int \mu(t) K^-(t, t_0, \rho)\, dt = f(t_0) \tag{2.4}$$

can also be studied in a similar manner.

3. An integral equation of the first kind of the form

$$\frac{1}{\pi i} \int \omega(t) L^+(t, t_0, \rho)\, dt = f(t_0), \quad -\rho < t_0 < \rho \tag{3.1}$$

$$L^{\pm}(t, t_0, \rho) = \frac{1}{t - t_0} \pm \frac{1}{t - \rho^2/t_0}$$

is of definite interest for applications.

Its solution is obviously given by the formula

$$\omega(t_0) = \frac{1}{\pi i} \int f(t) L^-(t, t_0, \rho)\, dt \tag{3.2}$$

or by its equivalent

$$\omega(t_0) = \frac{1}{\pi i} \int [f(t) - f(t_0)] L^-(t, t_0, \rho)\, dt$$

from which the continuity of the desired density $\omega(t)$ is clear.

However, this value of $\omega(t)$ is the solution of (3.1) only for absolute conservation of the following condition to which the given function $f(t)$ should be subjected:

$$\frac{1}{\pi i} \int f(t) \ln \frac{\rho - t}{\rho + t} \frac{dt}{t} = 0 \quad \left(\int \ln \frac{\rho - t}{\rho + t} \frac{dt}{t} = -\frac{\pi^2}{2} \right) \tag{3.3}$$

It is simplest of all to see the authenticity of the solution (3.2) by substituting the density it gives into the initial Eq.(1.1). In the more general case (in the presence of a parameter) (3.1) was examined in /4/. It is true that the direct passage for the parameter value $\lambda = 1$, at which (3.1) is obtained, is not evident, but utilization of (2.1) from /5/ also results in (3.2).

Remark 1. For $f(t) = t$ we obtain from (3.2)

$$\omega(t_0) = \left(t_0 - \frac{\rho^2}{t_0}\right) \left[\ln \frac{\rho - t_0}{-\rho - t_0} - \pi i \right]$$

As is seen, the density $\omega(t_0)$ is continuous on the closed segment γ_0.

We now set $f(t) = t^2$. We find by carrying out simple calculations

$$\omega(t_0) = \frac{2\rho}{\pi i}\left(t_0 - \frac{\rho^2}{t_0}\right) + \left(t_0^2 - \frac{\rho^4}{t_0^2}\right)\left[\frac{1}{\pi i} \ln \frac{\rho - t_0}{-\rho - t_0} - 1\right]$$

Taking into account further the expansion in the region of $z = 0$ (from the side $\operatorname{Im} z > 0$)

$$\ln \frac{\rho - t_0}{-\rho - t_0} = \pi i - 2\frac{t_0}{\rho} + O(t_0^3)$$

we can prove the continuity of $\omega(t_0)$ in the section γ_0.

Remark 2. For the singular equation of the form

$$\frac{1}{\pi i} \int \omega(t) L^-(t, t_0, \rho)\, dt = f(t_0) \tag{3.4}$$

the required solution is

$$\omega(t_0) = \frac{1}{\pi i} \int f(t) L^+(t, t_0, \rho)\, dt \tag{3.5}$$

For $f(t_0) = 1$ we have

$$\omega(t_0) = 2\left(\frac{1}{\pi i}\ln\frac{\rho - t_0}{-\rho - t_0} - 1\right) \tag{3.6}$$

Using the formulas

$$\frac{1}{\pi i}\int \ln\frac{\rho - t}{-\rho - t}\frac{dt}{t - t_0} = \frac{1}{2\pi i}\ln^2\frac{\rho - t_0}{-\rho - t_0}$$

$$\frac{1}{\pi i}\int \ln\frac{\rho - t}{-\rho - t}\frac{dt}{t - \rho^2/t_0} = \frac{1}{2\pi i}\ln^2\frac{\rho - t_0}{-\rho - t_0} - \frac{\pi i}{2}$$

we can prove that the function $\omega(t)$, given by (3.6), in fact satisfies Eq. (3.4) (for $f(t_0) = 1$).

4. We will now consider the singular integral equation which in its external structure hardly differs from (2.1), nevertheless, it does differ from it qualitatively. This equation is

$$\omega(t_0) + \frac{B}{\pi i}\int \omega(t) L^+(t, t_0, \rho)\, dt = f(t_0) \quad (-\rho < t_0 < \rho) \tag{4.1}$$

where $B \neq \pm 1$ is some constant. Proceeding as before we can write this equation in the form

$$\frac{1}{\pi i}\int \omega(t) L^+(t, t_0, \rho)\, dt = \mu(t_0) \tag{4.2}$$

$$\mu(t_0) = \frac{1}{B}[-\omega(t_0) + f(t_0)] \tag{4.3}$$

i.e. the free terms depends directly on the required density $\omega(t)$.
Inverting (4.2) using (3.2) we obtain

$$\omega(t_0) = \frac{1}{\pi i}\int \mu(t) L^-(t, t_0, \rho)\, dt \tag{4.4}$$

As has been established, this value of the density $\omega(t_0)$ will serve as a solution of Eq. (4.2) only if

$$\frac{1}{\pi i}\int \mu(t)\ln\frac{\rho - t}{\rho + t}\frac{dt}{t} = \frac{1}{\pi i}\int \mu(t)\left[\ln\frac{\rho - t}{-\rho - t} - \pi i\right]\frac{dt}{t} = 0 \tag{4.5}$$

We can show that this condition is satisfied by fixing an appropriate constant, if this exists in $f(t)$, and also in $\mu(t)$. We can prove this by establishing that

$$\int\left[\ln\frac{\rho - t}{-\rho - t} - \pi i\right]\frac{dt}{t} \neq 0 \tag{4.6}$$

The following relationship holds:

$$\frac{1}{2\pi i}\int\left(\frac{\rho - t}{-\rho - t}\right)^\alpha \frac{dt}{t - z} = \frac{1}{1 - e^{-2\pi i\alpha}}\left[\left(\frac{\rho - z}{-\rho - z}\right)^\alpha - 1\right] = \tag{4.7}$$

$$\left[\frac{1}{2\pi i\alpha} + \frac{1}{2} + O(\alpha)\right]\left[\alpha\ln\frac{\rho - z}{-\rho - z} + \frac{\alpha^2}{2}\ln^2\frac{\rho - z}{-\rho - z} + \ldots\right] =$$

$$\frac{1}{2\pi i}\ln\frac{\rho - z}{-\rho - z} + \alpha\left[\frac{1}{2}\ln\frac{\rho - z}{-\rho - z} + \frac{1}{4\pi i}\ln^2\frac{\rho - z}{-\rho - z}\right] + O(\alpha)^2$$

Comparing coefficients of terms without α in the previous integral relation (on the left and on the right), and then containing the first power of α, we find successively

$$\int\frac{dt}{t - z} = \ln\frac{\rho - z}{-\rho - z} \tag{4.8}$$

$$\frac{1}{2\pi i}\int \ln\frac{\rho - t}{-\rho - t}\frac{dt}{t - z} = \frac{1}{2}\ln\frac{\rho - z}{-\rho - z} + \frac{1}{4\pi i}\ln^2\frac{\rho - z}{-\rho - z}$$

Letting the variable z tend to zero, we obtain

$$\int \ln\frac{\rho - t}{-\rho - t}\frac{dt}{t} = -\frac{\pi^2}{2}$$

from which (4.6) follows.
In expanded form relationship (4.4) becomes

$$B\omega(t_0) + \frac{1}{\pi i}\int \omega(t) L^-(t, t_0, \rho)\, dt = \frac{1}{\pi i}\int f(t) L^-(t, t_0, \rho)\, dt$$

By suitably combining this equation with the initial Eq. (4.1), we again arrive at a Carleman equation

$$(1 + B^2)\,\omega(t_0) + \frac{2B}{\pi i}\int \frac{\omega(t)}{t - t_0}\,dt = g(t_0)$$
$$g(t_0) = f(t_0) + \frac{B}{\pi i}\int f(t)\,L^-(t, t_0, \rho)\,dt$$

The general solution of latter equation is obtained in the form of a sum of components

$$\begin{aligned} &\omega(t) = \omega_*(t) + \omega_0(t) \\ &\omega_*(t_0) = \frac{1+B^2}{1-B^2}\,g(t_0) - \frac{2B}{(1-B^2)^2}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\lambda}\frac{1}{\pi i}\int g(t)\,\times \\ &\left(\frac{\rho - t}{-\rho - t}\right)^{-\lambda}\frac{dt}{t - t_0} \\ &\omega_0(t) = C\,\frac{1}{\rho - t}\left(\frac{\rho - t}{-\rho - t}\right)^{\lambda} \end{aligned} \tag{4.9}$$

Here $\omega_*(t)$ is the solution of the inhomogeneous equation while $\omega_0(t)$ is the solution of the corresponding homogeneous equation, where C is some constant (to be determined later) and the parameter is

$$\begin{aligned} \lambda = \frac{1}{2\pi i}\ln\left(\frac{1-B}{1+B}\right)^2 = \frac{1}{\pi i}\ln\frac{1-B}{1+B} = \\ \frac{\alpha}{\pi} + \frac{1}{\pi i}\ln\left|\frac{1-B}{1+B}\right|, \quad |\alpha| < \pi \quad (B \neq \pm 1) \end{aligned} \tag{4.10}$$

On the basis of (4.3) and (4.9) we give the solvability condition for (4.1) the following form

$$\begin{aligned} C\,\frac{1}{\pi i}\int \frac{1}{\rho - t}\left(\frac{\rho - t}{-\rho - t}\right)^{\lambda}\left[\ln\frac{\rho - t}{-\rho - t} - \pi i\right]\frac{dt}{t} = \\ \frac{1}{\pi i}\int [-\omega_*(t) + f(t)]\left[\ln\frac{\rho - t}{-\rho - t} - \pi i\right]\frac{dt}{t} \end{aligned} \tag{4.11}$$

(it is clear that for real values of B it is necessary to set $\alpha = 0$). It will be established below that the integral in the last equality, which is a coefficient for the constant C, is different from zero. This enables us to determine the value of C that will ensure that condition (4.5) is satisfied and then enables us to calculate the density (4.9) which solves the initial Eq. (4.1).

We will now evaluate the integral

$$I(\lambda) = \frac{1}{2\pi i}\int \frac{1}{\rho - t}\left(\frac{\rho - t}{-\rho - t}\right)^{\lambda}\left[\ln\frac{\rho - t}{-\rho - t} - \pi i\right]\frac{dt}{t}$$

We separate this integral into a pair of individual integrals, in the number of components contained in the square brackets (each of them must be understood in the principal-value sense).

We evaluate the second of the named integrals by residue theory. We obtain

$$\frac{1}{2\pi i}\int \frac{1}{\rho - t}\left(\frac{\rho - t}{-\rho - t}\right)^{\lambda}\frac{dt}{t} = \frac{1}{2i\rho}\,e^{i\pi\lambda}\operatorname{ctg}\pi\lambda$$

Differentiating the last equality with respect to the parameter λ we obtain

$$I(\lambda) = -\frac{\pi}{2i\rho}\,\frac{e^{i\pi\lambda}}{\sin^2\pi\lambda}$$

As we see, the integral $I(\lambda)$ differs from zero, and therefore, the constant C can be determined from (4.11) (for the mentioned value (4.10) for the parameter λ).

Remark 3. The singular equations themselves of such a comparatively simplified structure

$$\frac{1}{\pi i}\int_0^\infty \mu(t)\,M^{\mp}(t, t_0)\,dt = f(t_0), \quad M^{\mp}(t, t_0) = \frac{1}{t - t_0} \mp \frac{1}{t + t_0} \tag{4.12}$$

are of interest.

Their solutions have the following respective forms:

$$\mu(t) = \frac{1}{\pi i}\int_0^\infty f(t_0)\,M^{\pm}(t, t_0)\,dt_0 \tag{4.13}$$

In order to see that the values of the density $\mu(t)$ given by the last formulas are actually the solutions of the singular Eqs.(4.12), the direct substitution of the densities (4.13) into the appropriate initial equations should be performed and satisfaction of the relationships

$$\frac{1}{\pi i}\int_0^\infty M^{\mp}(t, t_0)\, dt \frac{1}{\pi i}\int_0^\infty f(t_1)\, M^{\pm}(t_1, t)\, dt_1 = f(t_0)$$

should actually be verified.

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ON AN INTEGRAL EQUATION OF THE PROBLEM FOR AN ELASTIC STRIP WITH A SLIT*

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A new singular integral equation is obtained that describes the elastic equilibrium of a strip with both an inner and an edge slit (crack) and has a considerable advantage over existing equations /1-9/, etc.) from the viewpoint of a numerical realization and clarification of the analytical relationship with an analogous equation for a half-plane. Numerical results are given of a computation of the stress intensity coefficients at the tips of the inner and edge cracks that refine data in the literature.

1. Let an elastic body occupy the strip $0 < y < H$, $-\infty < x < \infty$ with a rectilinear slit along the Oy axis between the points $y = a$, $y = b$, $a \geqslant 0, b \leqslant H$. The strip boundary is stress-free, while the stresses $\sigma_x = p(y)$, $\tau_{xy} = 0$ are given on the slit edges. Then the state of stress of the body under consideration is described /10/ by using two regular functions of the complex variable $z = x + iy$:

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}], \quad \sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{1.1}$$

that satisfy the boundary conditions on the slit edges

$$\Phi^{\pm}(iy) + \overline{\Phi^{\pm}(iy)} + iy\Phi'^{\pm}(iy) - \Psi^{\pm}(iy) = p(y) \tag{1.2}$$

and on the strip boundary

$$\begin{gathered}\Phi(x) + \overline{\Phi(x)} + x\Phi'(x) + \Psi(x) = 0 \\ \Phi(x + iH) + \overline{\Phi(x + iH)} + (x - iH)\Phi'(x + iH) + \\ \Psi(x + iH) = 0\end{gathered} \tag{1.3}$$

Values of the functions on the left and right edges of the slit and marked by plus and

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